# GLOBALLY GENERATED VECTOR BUNDLES OF RANK 2 ON A SMOOTH QUADRIC THREEFOLD

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Abstract. We investigate the existence of globally generated vector bundles of rank 2 with $c_1 \le 3$ on a smooth quadric threefold and determine their Chern classes. As an automatic consequence, every rank 2 globally generated vector bundle on $Q$ with $c_1 = 3$ is an odd instanton up to twist.

## 1. Introduction

Globally generated vector bundles on projective varieties play an important role in algebraic geometry. If they are non-trivial they must have strictly positive first Chern class. Globally generated vector bundles on projective spaces with low first Chern class have been investigated in several papers. If $c_1(\mathcal{E}) = 1$ then it is easy to see that modulo trivial summands we have only $\mathcal{O}_{\mathbb{P}^n}(1)$ and $T\mathbb{P}^n(-1)$. The classification of rank $r$ globally generated vector bundles with $c_1 = 2$ is settled in [14]. In [9] the second author carried out the case of rank two with $c_1 = 3$ on $\mathbb{P}^3$ and in [3] the authors continued the study until $c_1 \le 5$. This classification was extended to any rank in [10] and to any $\mathbb{P}^n$ $(n \ge 3)$ in [1] and [15]. In [5] are shown the possible Chern classes of rank two globally generated vector bundles on $\mathbb{P}^2$.

Let $Q$ be a smooth quadric threefold over an algebraically closed field of characteristic zero. The aim of this paper is to investigate the existence of globally generated vector bundles of rank 2 on $Q$ with $c_1 \le 3$. We prove the following theorem:

**Theorem 1.1.** *There exists an indecomposable and globally generated vector bundle of rank 2 on $Q$ with the Chern classes $(c_1, c_2)$, $c_1 \le 3$ if and only if*

$$(c_1, c_2) \in \{(1,1), (2,4), (3,5), (3,6), (3,7), (3,8), (3,9)\}.$$

We use an old method of associating to a rank 2 vector bundle on $Q$ a curve in $Q$, and relate properties of the bundle to properties of the curve. In Sect.2 we set up basic computations and deal with the case $c_1 = 1$ as a preliminary case. In Sect.3 we prove that every indecomposable and globally

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generated vector bundle of rank 2 on $Q$ with $c_1 = 2$ is a pull-back of a null-correlation bundle on $\mathbb{P}^3$ twisted by 1. In Sect.4, 5 and 6, we deal with the case of $c_1 = 3$. First we determine the possible second Chern class $c_2$ using the Liaison theory, that is $5 \le c_2 \le 9$. In each case we prove the existence of globally generated vector bundle of rank 2. In Sect.4 we explain the case of $c_2 = 5, 6, 7$ based on the result [12] about the moduli spaces of rank 2 vector bundles on $Q$. The critical part of this paper is on the existence of globally generated vector bundles of rank 2 with $c_2 = 8$ and 9. It is equivalent to the existence of a smooth elliptic curve $C$ of degree $c_2$ whose ideal sheaf twisted by 3 is globally generated. The main ingredient is the result in [7] with replacing $\mathbb{P}^3$ by $Q$, by which we can deform a nodal reducible curve of degree $c_2$ constructed in a suitable way to a smooth elliptic curve that we need.

A typical way to construct a vector bundle on $Q$ is by restriction of a vector bundle on $\mathbb{P}^4$ or by a pull-back of a vector bundle on $\mathbb{P}^3$ along a linear projection from $Q$ to $\mathbb{P}^3$. The spinor bundle $\Sigma$ is not obtained by either one of these ways and it plays an important role in describing the globally generated vector bundle on $Q$. In fact, from the classification we observe that every rank two globally generated vector bundle on $Q$ with $c_1 = 3$ is, up to twist, an odd instanton that is the cohomology of a monad involving $\Sigma$ (see [6]).

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## 2. Preliminaries

Let $Q$ be a smooth quadric hypersurface in $\mathbb{P}^4$ and let $\mathcal{E}$ be a coherent sheaf of rank $r$ on $Q$. Then we have:

$$\begin{aligned}
c_1(\mathcal{E}(k)) &= c_1 + kr \\
c_2(\mathcal{E}(k)) &= c_2 + 2k(r-1)c_1 + 2k^2\binom{r}{2} \\
c_3(\mathcal{E}(k)) &= c_3 + k(r-2)c_2 + 2k^2\binom{r-1}{2} + 2k^3\binom{r}{3} \\
\chi(\mathcal{E}) &= (2c_1^3 - 3c_1c_2 + 3c_3)/6 + 3(c_1^2 - c_2)/2 + 13c_1/6 + r,
\end{aligned}$$

where $(c_1, c_2, c_3)$ is the Chern classes of $\mathcal{E}$. In particular, when $\mathcal{E}$ is a vector bundle of rank 2 with $c_1 = -1$, we have

$$\chi(\mathcal{E}) = 1 - c_2\ ,\ \chi(\mathcal{E}(1)) = 6 - 2c_2\ ,\ \chi(\mathcal{E}(-1)) = 0,$$
$$\chi(\mathcal{E}nd(\mathcal{E})) = 7 - 6c_2.$$

Let $\mathfrak{M}(c_1, c_2)$ be the moduli space of stable vector bundles of rank 2 on $Q$ with the Chern classes $(c_1, c_2)$.

**Proposition 2.1.** [13] *Let $\mathcal{E}$ be a globally generated vector bundle of rank $r$ on $Q$ such that $H^0(\mathcal{E}(-c_1)) \neq 0$, where $c_1$ is the first Chern class of $\mathcal{E}$.*

*Then we have*

$$\mathcal{E} \simeq \mathcal{O}_Q^{\oplus(r-1)} \oplus \mathcal{O}_Q(c_1).$$

Now assume that $\mathcal{E}$ is a globally generated vector bundle of rank 2 on $Q$. Then $\mathcal{E}$ admits an exact sequence

$$0 \to \mathcal{O}_Q \to \mathcal{E} \to \mathcal{I}_C(c_1) \to 0, \tag{1}$$

where $C$ is a smooth curve on $Q$. Notice that $\omega_C \simeq \mathcal{O}_C(-3 + c_1)$ and $c_2(\mathcal{E}) = \deg(C)$.

If $l$ is a line on $Q$, then $\mathcal{E}|_l$ is also globally generated, which means in particular that $c_1(\mathcal{E}) \geq 0$.

**Remark 2.2.** $\mathcal{E} \cong \mathcal{O}_Q(c_1) \oplus \mathcal{O}_Q$ if and only if $C = \emptyset$.

**Proposition 2.3.** *We have $c_1(\mathcal{E}) = 1$ if and only if either $\mathcal{E} \cong \mathcal{O}_Q(1) \oplus \mathcal{O}_Q$ or $\mathcal{E}$ is the spinor bundle.*

*Proof.* Both $\mathcal{O}_Q(1) \oplus \mathcal{O}_Q$ and the spinor bundle are globally generated and have the prescribed Chern classes. Hence it is sufficient to consider the case when $C \neq \emptyset$ by the remark 2.2. Since $C$ is smooth and $\mathcal{I}_C(1)$ is globally generated, it is contained in a codimension 2 linear section of $Q$. Hence $C$ is either a smooth conic or a line. In both cases $C$ is ACM. From the sequence (1) we get that $\mathcal{E}$ is an ACM vector bundle. Hence $\mathcal{E}$ is either decomposable or a twist of the spinor bundle by Theorem 3.5 in [11]. Since $\mathcal{E}$ is globally generated and $c_1(\mathcal{E}) = 1$, we get that either $\mathcal{E} \cong \mathcal{O}_Q(1) \oplus \mathcal{O}_Q$ or $\mathcal{E}$ is the spinor bundle. □

## 3. Rank two globally generated bundles with $c_1 = 2$

In this section we prove the following result.

**Proposition 3.1.** *Let $\mathcal{E}$ be a rank 2 vector bundle with $c_1(\mathcal{E}) = 2$. $\mathcal{E}$ is globally generated if and only if $\mathcal{E}$ is either isomorphic to*

(1) *$\mathcal{O}_Q(a) \oplus \mathcal{O}_Q(2-a)$ with $a = 0, 1$ or*
(2) *a pull-back of a null-correlation bundle on $\mathbb{P}^3$ twisted by 1*

**Lemma 3.2.** *Let $C = E_1 \sqcup E_2 \sqcup J \subset Q$ be a curve with $E_1$ and $E_2$ smooth conics and $J \neq \emptyset$. Then $\mathcal{I}_C(2)$ is not globally generated.*

*Proof.* Assume that $\mathcal{I}_C(2)$ is globally generated. Let $H \subset \mathbb{P}^4$ denote a general hyperplane containing $E_1$. Set $Q' := H \cap Q$. $Q'$ is a smooth quadric surface, the scheme $Z := H \cap (E_2 \cup J)$ is a zero-dimensional scheme of degree at least 3 and $Z \cap E_1 = \emptyset$. Since $\mathcal{I}_C(2)$ is globally generated, so is $\mathcal{I}_{E_1 \cup Z, Q'}(2,2)$; since $E_1$ is a curve of type $(1,1)$ on $Q'$ and $Z \cap E_1 = \emptyset$, so $\mathcal{I}_Z(1,1)$ is globally generated. But this is absurd since $\deg(Z) \geq 3$. □

*Proof of Proposition 3.1.* Let us assume that the vector bundle $\mathcal{E}$ on $Q$ is globally generated with $c_1(\mathcal{E}) = 2$ and then it fits into the exact sequence (1) with $c_1 = 2$ and $C$ a smooth curve. Since $\omega_C \simeq \mathcal{O}_C(-1)$, $C$ is a disjoint union of conics, i.e. $C = C_1 \cup \cdots \cup C_r$, where each $C_i$ is a conic.

By Remark 2.2 we have $\mathcal{E} \cong \mathcal{O}_Q(2) \oplus \mathcal{O}_Q$ if and only if $r = 0$. Now assume $r > 0$. Lemma 3.2 gives $r \in \{1, 2\}$. As the first case let us assume that $r = 1$. Since a smooth conic is ACM, the sequence (1) gives that $\mathcal{E}$ is ACM. Hence we have $\mathcal{E} \cong \mathcal{O}_Q(1) \oplus \mathcal{O}_Q(1)$ by Theorem 3.5 in [11].

Now let us assume that $r = 2$. Let $M_i \subset \mathbb{P}^4$ be the plane spanned by $C_i$. Since $M_i \cap Q = C_i$ and $C_1 \cap C_2 = \emptyset$, the set $M_1 \cap M_2$ cannot be a line. Hence $M_1 \cap M_2$ is a point, say $P$. Since $M_i \cap Q = C_i$ and $C_1 \cap C_2 = \emptyset$, we have $P \notin Q$. The linear projection $\ell_P : \mathbb{P}^4 \setminus \{P\} \longrightarrow \mathbb{P}^3$ send each $E_i$ onto a line $L_i$. In Prop. 3.2 of [16], it was shown that all bundles as an extension in (1) are pull-back from bundles from extensions

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_{L_1 \sqcup L_2}(2) \longrightarrow 0 \tag{2}$$

on $\mathbb{P}^3$, in which $\mathcal{F}$ is a null-correlation bundle twisted by 1.

## 4. Rank two globally generated bundles with $c_1 = 3$

Let $\mathcal{E}$ be a globally generated vector bundle of rank 2 on $Q$ with the Chern classes $(c_1, c_2)$, $c_1 \geq 3$. It fits into the exact sequence (1).

**Lemma 4.1.** *The Chern classes $(c_1, c_2)$ of $\mathcal{E}$ satisfies the following inequality for $c_1 \geq 3$:*

$$c_2 \leq \frac{2}{3}(2c_1 + 1)(c_1 - 1).$$

*In particular, if $c_1 = 3$, we have $c_2 \leq 9$.*

*Proof.* Since $\mathcal{I}_C(c_1)$ is globally generated, so there are two hypersurfaces of degree $c_1$ in $Q$ whose intersection is a curve $X$ containing $C$. Let $Y$ be a curve such that $X = C + Y$. If $\mathcal{E}$ does not split, then $Y$ is not empty so we have the exact sequence of liaison:

$$0 \to \mathcal{I}_X(c_1) \to \mathcal{I}_C(c_1) \to \omega_Y(3 - c_1) \to 0.$$

Since $\mathcal{I}_C(c_1)$ is globally generated, so is $\omega_Y(3 - c_1)$. It implies that

$$\deg(\omega_Y(3 - c_1)) = 2g' - 2 + d'(3 - c_1) \geq 0,$$

$(d' = \deg(Y), g' = g(Y))$ and so $g' \geq 0$. On the other hand, by liaison, we have

$$g' - g = \frac{1}{2}(d' - d)(2c_1 - 3)$$

and $d' = 2c_1^2 - d$, $2g - 2 = d(c_1 - 3)$ since $\omega_C(3 - c_1) = \mathcal{O}_C$. Here $d = \deg(C)$ and $g = g(C)$. Thus we have $g' = 1 + \frac{d(c_1-3)}{2} + (c_1^2 - d)(2c_1 - 3) \geq 0$ and so

$$c_2 \leq \frac{2(2c_1^3 - 3c_1^2 + 1)}{3c_1 - 3} = \frac{2}{3}(2c_1 + 1)(c_1 - 1).$$

$\square$

Assume that $\mathcal{E}$ is a globally generated vector bundle of rank 2 on $Q$ with $c_1 = 3$ that fits into the sequence:

$$0 \to \mathcal{O}_Q \to \mathcal{E} \to \mathcal{I}_C(3) \to 0,$$

where $C$ is a smooth curve with $\deg(C) = c_2(\mathcal{E})$.

Note that if $H^0(\mathcal{E}(-3)) \neq 0$, then $\mathcal{E}$ is isomorphic to $\mathcal{O}_Q \oplus \mathcal{O}_Q(3)$, which is globally generated. So let us assume that $H^0(\mathcal{E}(-3)) = 0$.

As the first case, let us assume that $H^0(\mathcal{E}(-2)) \neq 0$, i.e. $\mathcal{E}$ is unstable.

**Proposition 4.2.** *Let $\mathcal{E}$ be a globally generated unstable vector bundle of rank 2 on $Q$ with $c_1(\mathcal{E}) = 3$. Then $\mathcal{E}$ is a direct sum of line bundles. In other words, we have*

$$\mathcal{E} \simeq \mathcal{O}_Q(a) \oplus \mathcal{O}_Q(3-a)\ ,\ \ a = 0, 1.$$

*Proof.* Note that $h^0(\mathcal{I}_C(1)) \geq 0$ and so $C$ is contained in the complete intersection of $Q$ and two hypersurfaces of degree 1 and 3 inside $\mathbb{P}^4$ that is the ambient space containing $Q$. In particular, we have $\deg(C) \leq 6$. On the other hand, from a section in $H^0(\mathcal{E}(-2))$, we have a sequence

$$0 \to \mathcal{O}_Q(2) \to \mathcal{E} \to \mathcal{I}_{C'}(1) \to 0.$$

If $C'$ is empty, then $\mathcal{E}$ is isomorphic to $\mathcal{O}_Q(1) \oplus \mathcal{O}_Q(2)$. If not, the degree of $C'$, which is $c_2(\mathcal{E}) - 4$ is at least 1. Thus $\deg(C) = c_2(\mathcal{E})$ is either 5 or 6. Note that $\omega_C \simeq \mathcal{O}_C$ and so $C$ consists of smooth elliptic curves. If $\deg(C) = 5$, then $C$ is a quintic elliptic curve contained in a hyperplane section of $Q$ which is a quadric surface $Q_2$. In the case when $Q_2$ is smooth, let $(a, b)$ is the bidegree of $C$ as a divisor of the quadric surface and then we have $\deg(C) = 5 = a+b$ and $g(C) = 1 = ab-a-b+1$. But it is impossible. Similarly we can show that $\deg(C) = 6$ is not possible. If $Q_2$ is a quadric surface cone, we can show that it is impossible by Exercise V.2.9 of [8]. $\square$

Assume now that $H^0(\mathcal{E}(-2)) = 0$, i.e. $\mathcal{E}$ is stable. By the Bogomolov inequality, we have $c_2(\mathcal{E}) \geq 5$. Recall that $c_2(\mathcal{E}) \leq 9$.

Note that $\mathcal{E}$ fits into the sequence

$$0 \to \mathcal{O}_Q(1) \to \mathcal{E} \to \mathcal{I}_Z(2) \to 0. \tag{3}$$

with $Z$ a locally complete intersection, $\deg(Z) = c_2 - 4$ and $\omega_Z \simeq \mathcal{O}_Z(-2)$.

If $c_2(\mathcal{E}) = 5$, then $Z$ is a line. Form the (3) we get that $\mathcal{E}$ is ACM. Since $\mathcal{E}$ is stable, we get $\mathcal{E} \simeq \Sigma(1)$ [11]. Obviously $\Sigma(1)$ is globally generated.

If $c_2(\mathcal{E}) = 6$, then $\mathcal{E}$ is the cohomology of the following monad (Remark 4.8 in [12]):

$$0 \to \mathcal{O}_Q(1) \to \Sigma(1)^{\oplus 2} \to \mathcal{O}_Q(2) \to 0.$$

In this case, $Z$ is either 2 disjoint lines on $Q$ or a line with multiplicity 2. It is also known to be globally generated.

So we can assume that $H^0(\mathcal{E}(-1)) = 0$ and $c_2(\mathcal{E}) \geq 7$.

**Remark 4.3.** If $c_2(\mathcal{E}) = 7$, then a general vector bundle $\mathcal{E}$ in $\mathfrak{M}(3,7)$ can be shown to be globally generated using the 'Castelnuovo-Mumford criterion' (Theorem 5.2 in [12]). The same result also follows from the Lemma 6.2 with $C = A$. Since $h^0(\mathcal{I}_C(2)) = 0$ and $h^0(C, \mathcal{O}_C(2)) = 14 = h^0(\mathcal{O}_Q(2))$, (1) gives $h^1(\mathcal{E}(-1)) = 0$. Since $C$ is an elliptic curve, we have $h^1(C, \mathcal{O}_C(1)) = 0$. Hence (1) gives $h^2(\mathcal{E}(-2)) = h^1(C, \mathcal{O}_C(1)) = 0$. We have $h^3(\mathcal{E}(-3)) = h^0(\mathcal{E}^\vee) = h^0(\mathcal{E}(-3)) = 0$. Hence the 'Castelnuovo-Mumford criterion' gives that $\mathcal{E}$ is globally generated.

## 5. Case $(c_1, c_2) = (3, 8)$

Let $\mathcal{F} = \mathcal{E}(-2)$. We can compute:

$$\begin{cases} H^1(\mathcal{F}(t)) = 0 \text{ for } t \leq -1 \\ \chi(\mathcal{F} \otimes \mathcal{F}^\vee) = -17 \\ \chi(\mathcal{F}) = -\chi(\mathcal{F}(-2)) = -3 \\ \chi(\mathcal{F}(1)) = -\chi(\mathcal{F}(-3)) = -2 \\ \chi(\mathcal{F}(-1)) = 0 \\ \chi(\mathcal{F}(2)) = 7 \end{cases}$$

Then the cohomology table for $\mathcal{F} = \mathcal{E}(-2)$ is as follows:

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 3 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 3 | 2 | $b$ |
| 0 | 0 | 0 | 0 | 0 | $a$ |
| -3 | -2 | -1 | 0 | 1 | 2 |

with $a - b = 7$.

**Proposition 5.1.** *The vector bundles $\mathcal{E}$ in $\mathfrak{M}(3,8)$ with $H^0(\mathcal{E}(-1)) = 0$ is the cohomology of the following monad:*

$$0 \longrightarrow \mathcal{O}_Q(1)^{\oplus 3} \longrightarrow \Sigma(1)^{\oplus 4} \longrightarrow \mathcal{O}_Q(2)^{\oplus 3} \longrightarrow 0$$

*Proof.* Let us consider the sequence killing $H^1(\mathcal{F})$:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{B} \longrightarrow \mathcal{O}_Q^{\oplus 3} \longrightarrow 0 \tag{4}$$

$H^2_*(\mathcal{F}) \cong H^2_*(\mathcal{B})$ and $H^1(\mathcal{F}) = H^0(\mathcal{F}) = 0$ since the map $H^0(\mathcal{O}_Q^{\oplus 3}) \longrightarrow H^1(\mathcal{F})$ is an isomorphism. Moreover $h^3(\mathcal{B}(-3)) = h^3(\mathcal{O}_Q(-3)^{\oplus 3}) = 3$. So the cohomology table for $\mathcal{B}$ and $\mathcal{B}^\vee$ are as follows respectively :

| | | | |
|---|---|---|---|
| 3 | 0 | 0 | 0 |
| 2 | 3 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| -3 | -2 | -1 | 0 |

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 3 | 2 |
| 0 | 0 | 0 | 3 |
| -3 | -2 | -1 | 0 |

Now let us consider the sequence killing $H^1(\mathcal{B}^\vee)$ and $H^1(\mathcal{B}^\vee(-1))$:

$$0 \longrightarrow \mathcal{B}^\vee \longrightarrow \mathcal{P} \longrightarrow \mathcal{O}_Q(1)^{\oplus 3} \oplus \mathcal{O}_Q^{\oplus 2} \longrightarrow 0 \tag{5}$$

$H^2_*(\mathcal{P}) \cong H^2_*(\mathcal{B}^\vee)$ and $H^1(\mathcal{P}) = H^1(\mathcal{P}(-1)) = 0$. $h^0(\mathcal{B}^\vee) - h^0(\mathcal{P}) + h^0(\mathcal{O}_Q(1)^{\oplus 3} \oplus \mathcal{O}_Q^{\oplus 2}) - h^1(\mathcal{B}^\vee) = 0$ then $h^0(\mathcal{P}) = 18$. Moreover $h^3(\mathcal{P}(-3)) = h^3(\mathcal{O}_Q(-3)^{\oplus 2}) = 2$. So the cohomology table for $\mathcal{P}$ and $\mathcal{P}^\vee$ are as follows:

| | | | |
|---|---|---|---|
| 2 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 18 |
| -3 | -2 | -1 | 0 |

| | | | |
|---|---|---|---|
| 18 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 2 |
| -3 | -2 | -1 | 0 |

Since $\mathcal{P}(1)$ and $\mathcal{P}^\vee(1)$ are Castelnuovo-Mumford regular, so $\mathcal{P}$ is an ACM vector bundle. Note that the rank of $\mathcal{P}$ is 10, $c_1(\mathcal{P}) = 4$, $h^0(\mathcal{P}) = 18$ and $h^0(\mathcal{P}^\vee) = 2$. So we have $\mathcal{P} = \mathcal{O}_Q^{\oplus 2} \oplus \Sigma^{\oplus 4}$, the sequence (5) reduces to

$$0 \longrightarrow \mathcal{B}^\vee \longrightarrow \Sigma^{\oplus 4} \longrightarrow \mathcal{O}_Q(1)^{\oplus 3} \longrightarrow 0$$

and the monad is as we claimed.

□

**Remark 5.2.** In [6], every vector bundle $\mathcal{F}$ of rank 2 with the Chern classes $(c_1, c_2) = (-1, k)$ and $H^1(\mathcal{F}(-1)) = 0$ on $Q$, can be shown to be the cohomology of a monad

$$0 \to \mathcal{O}_Q^{\oplus k-1} \to \Sigma^{\oplus k} \to \mathcal{O}_Q(1)^{\oplus k-1} \to 0$$

using the Kapranov spectral sequences on $Q$.

Note that $H^1(\mathcal{E}(-3)) = H^1(\mathcal{I}_C) = 0$ implies that $H^0(\mathcal{O}_C) = 1$, i.e. $C$ is a smooth irreducible elliptic curve of degree 8.

**Proposition 5.3.** *There is a globally generated and stable vector bundle $\mathcal{E}$ of rank 2 on $Q$ with $(c_1, c_2) = (3, 8)$ such that*

- $h^0(\mathcal{E}(-1)) = 0$, $h^0(\mathcal{E}) = 7$
- $h^1(\mathcal{E}(-1)) = 2, h^1(\mathcal{E}(-2)) = 3$ *and* $h^1(\mathcal{E}(t)) = 0$ *for all* $t \geq 0, t \leq -3$

Let us fix 5 distinct points $P_1, \dots P_5 \in \mathbb{P}^2$ such that no 3 of them are collinear. Let $\pi : W \longrightarrow \mathbb{P}^2$ be the blowing-up of these 5 points. The anticanonical line bundle $\omega_W^\vee$ of $W$ is very ample and the image $U \subset \mathbb{P}^4$ of $W$ by the complete linear system $|\omega_W^\vee|$ is a smooth Del Pezzo surface of degree 4 which is the complete intersection of two quadric hypersurfaces ([4]) (in the standard notation for Del Pezzo surfaces $\omega_W^\vee = (3; 1, 1, 1, 1, 1)$, i.e. it is given by the strict transform of all cubic plane curves containing all points $P_1, \dots, P_5$). Conversely, any smooth complete intersection $U' \subset \mathbb{P}^4$ is the complete intersection of two quadric hypersurfaces. Hence for general $P_1, \dots, P_5$ we may assume that $U$ is contained in a smooth quadric. Hence we see $U \subset Q$. Every curve $C \subset U$ is a curve contained in $Q$. We have

$h^0(U, \mathcal{O}_U(1)) = 5$, $h^0(U, \mathcal{O}_U(2)) = 13$ and $h^0(U, \mathcal{O}_U(3)) = \chi(O_U) + \mathcal{O}_U(3) \cdot \mathcal{O}_U(4)/2 = 25$ by the Riemann-Roch theorem.

**Remark 5.4.** Let $E \subset \mathbb{P}^4$ be a smooth and non-degenerate elliptic curve such that $\deg(E) = 6$. Since $h^1(\mathcal{I}_{E,\mathbb{P}^4}(1)) = 1$, it is easy to check that $h^1(\mathcal{I}_{E,\mathbb{P}^4}(2)) = 0$. Castelnuovo-Mumford's lemma implies that $\mathcal{I}_{E,\mathbb{P}^4}(3)$ is spanned. Hence for every smooth elliptic curve $E \subset Q$ with $\deg(E) = 6$ and $E$ not contained in a hyperplane, the sheaf $\mathcal{I}_E(3)$ on $Q$ is spanned.

Fix any such smooth elliptic curve $E \subset Q$ such that $\deg(E) = 6$ and $E$ is not contained in a hyperplane of $\mathbb{P}^4$. Since $h^0(E, \mathcal{O}_E(2)) = 12 = h^0(\mathcal{O}_Q(2)) - 2$, it is contained in some quadric hypersurface of $Q$. Let $U \subset Q$ be the smooth Del Pezzo surface of degree 4 just introduced. We find smooth elliptic curves of degree 6 inside $U$ by taking the smooth elements of type $(3; 1, 1, 1, 0, 0)$. Fix one such curve $E$. Since $h^0(U, \mathcal{O}_U(2)) = 13$ and $h^0(E, \mathcal{O}_E(2)) = 12$, $E$ is contained in at least one quadric hypersurface, $T$, of $U$. See $E$ as a curve of type $(3; 1, 1, 1, 0, 0)$. $T$ has type $(6; 2, 2, 2, 2, 2)$. Hence the curve $T - E$ is a curve of type $(3; 1, 1, 1, 2, 2)$. No cubic plane curve with at least two singular points is integral. We see that $h^0(U, \mathcal{O}_U(T-U)) = 1$ and that the unique curve in $|T-U|$ is the disjoint union of two lines $R_1$ and $R_2$ with $R_1$ the image of the strict transform in $W$ of the only conic containing the five points $P_1, \dots, P_5$ (i.e. the only curve of type $(2; 1, 1, 1, 1, 1)$), while $R_2$ is the image in $U$ of the strict transform of the line of $\mathbb{P}^2$ spanned by $P_4$ and $P_5$ (i.e. the only curve of type $(1; 0, 0, 0, 1, 1)$).

**Remark 5.5.** Since $\mathcal{I}_E(3)$ is spanned the line bundle $\mathcal{L} := \mathcal{O}_U(3)(-E)$ is spanned. Let $\alpha : U \longrightarrow \mathbb{P}^r$, $r := h^0(U, \mathcal{L}) - 1$, denote the morphism induced by $|\mathcal{L}|$. Since $\mathcal{L}$ has type $(6; 2, 2, 2, 3, 3)$, we have $\mathcal{L} \cdot \mathcal{L} = 36-4-4-4-9-9 = 6$ and $\mathcal{L} \cdot \omega_U^\vee = \mathcal{L} \cdot \mathcal{O}_U(1) = 18 - 2 - 2 - 2 - 3 - 3 = 6$. Riemann-Roch gives $\chi(\mathcal{L}) = 1 + (6+6)/2 = 7$. Since $\mathcal{L}$ is spanned, Serre duality gives $h^2(\mathcal{L}) = 0$. Hence $r \geq 6$. Since $\alpha(U)$ spans $\mathbb{P}^r$, we have $\deg(\alpha(U)) \geq 6$. Since $\mathcal{L} \cdot \mathcal{L} > 0$, $\alpha(U)$ is a surface. Since $\deg(\alpha) \cdot \deg(\alpha(U)) = \mathcal{L} \cdot \mathcal{L} = 6$, we have $\deg(\alpha) = 1$, i.e. $\alpha$ is birational onto its image.

**Lemma 5.6.** *If the pair $(O_1, O_2)$ is general in $U \times U$, then $\mathcal{I}_{E \cup \{O_1, O_2\}, U}(3)$ is spanned.*

*Proof.* Take $\mathcal{L}$, $\alpha$ and $\mathbb{P}^r$ as in Remark 5.5. Lemma 5.6 just says that $\{O_1, O_2\}$ is the scheme-theoretic base locus, $\beta$, of the linear system $|\mathcal{I}_{\{O_1, O_2\}} \otimes \mathcal{L}|$ on $U$. Since $\alpha$ is birational onto its image and $O_1, O_2$ are general, we have $\alpha(O_1) \neq \alpha(O_2)$ and $\alpha^{-1}(\alpha(O_i)) = O_i$ as schemes. In characteristic zero a general codimension 2 section of the non-degenerate surface $\alpha(U)$ is in linearly general position ([2], pages 112–113). Since the pair $(\alpha(O_1), \alpha(O_2))$ is general in $\alpha(U) \times \alpha(U)$, we get that $\{\alpha(O_1), \alpha(O_2)\}$ is the scheme-theoretic intersection of $\alpha(U)$ with the line of $\mathbb{P}^r$ spanned by $\alpha(O_1)$ and $\alpha(O_2)$. Hence $\beta = \{O_1, O_2\}$. $\square$

Fix $E \subset U \subset Q$ as above and a general $(O_1, O_2) \in U \times U$. The union of the set of all lines containing $O_i$ and contained in $Q$ is the quadric cone

$(T_{O_i}Q) \cap Q$. Hence there is a line $D_i \subset Q$ containing $O_i$ and intersecting $E$. For general $(O_1, O_2)$ we may assume $D_1 \cap D_2 = \emptyset$ and that $D_i$ intersects quasi-transversally $E$ and only at the point $A_i := D_i \cap E$. Hence $Y := E \cup D_1 \cup D_2$ is a nodal and connected curve of degree 8 inside $Q$ with $p_a(Y) = 1$.

The reduced curve $Y$ is locally a complete intersection and hence its normal sheaf $\mathcal{N}_Y$ is a rank 2 vector bundle of degree $\deg(TQ|_Y) = 24$. Since $Y$ has nodes only at $A_1$ and $A_2$, the vector bundles $\mathcal{N}_Y|_E$ are obtained from $\mathcal{N}_E$ making two positive elementary transformations. Since $h^1(E, \mathcal{N}_E) = 0$, we have $h^1(E, \mathcal{N}_Y|_E) = 0$. Each normal bundle $\mathcal{N}_{D_i}$ is a direct sum of a degree 0 line bundle and a degree 1 line bundle. Hence for every vector bundle $\mathcal{F}$ on $D_i$ obtained from $\mathcal{N}_{D_i}$ making one negative elementary transformation has no factor of degree $\leq -2$. Hence $h^1(D_i, \mathcal{F}) = 0$. Hence $h^1(D_1 \cup D_2, \mathcal{R}) = 0$, where $\mathcal{R}$ is the vector bundle obtained from $\mathcal{N}_{D_1 \cup D_2}$ making the two negative elementary transformations at $A_1$ and $A_2$ associated to the tangent lines of $E$ at these points. Hence $h^1(Y, \mathcal{N}_Y) = 0$ and $Y$ is smoothable inside $Q$ (use [7], Theorem 4.1, for $Q$ instead of the smooth 3-fold $\mathbb{P}^3$). We get that the nearby smooth curves, $C$, have $h^1(\mathcal{N}_C) = 0$ and they form a 24-dimensional family smooth at $C$. By semicontinuity the general such $C$ has also $h^1(\mathcal{I}_C(3)) = 0$. Since in a flat family $\{C_\lambda\}$ of family with constant $h^1(\mathcal{I}_{C_\lambda}(3))$, the condition "$\mathcal{I}_{C_\lambda}(3)$ is spanned" is an open condition, to find a degree 8 smooth elliptic curve $C \subset Q$ with $\mathcal{I}_C(3)$ spanned (and hence to complete the case $c_1 = 3$, $c_2 = 8$), it is sufficient to prove that $\mathcal{I}_Y(3)$ is spanned.

**Lemma 5.7.** *The sheaf $\mathcal{I}_Y(3)$ is spanned.*

*Proof.* Let $\mathcal{B}$ denote the scheme-theoretic base-locus of the linear system $|\mathcal{I}_Y(3)|$ on $Q$. We need to prove that $\mathcal{B} = Y$ as schemes. Since $Y \cap U = E \cup \{O_1, O_2\}$ as schemes and $h^1(\mathcal{I}_U(3)) = h^1(\mathcal{O}_Q(1)) = 0$, $\mathcal{B}|_U$ is the scheme-theoretic base locus of the linear system $|\mathcal{I}_{\{O_1,O_2\}} \otimes \mathcal{L}|$ on $U$. Lemma 5.6 gives $\mathcal{B}|_U = E \cup \{O_1, O_2\}$ as schemes. Let $H \subset \mathbb{P}^4$ be the hyperplane spanned by the lines $D_1$ and $D_2$. For general $O_1, O_2$ we may assume that $Q_2 := Q \cap H$ is a smooth quadric surface. Since $U \cup Q_2 \in |\mathcal{I}_Y(3)|$, $h^1(\mathcal{I}_{Q_2}(3)) = h^1(\mathcal{O}_Q(2)) = 0$ and $\mathcal{B}|_U = E \cup \{O_1, O_2\}$, to prove the lemma it is sufficient to prove that $D_1 \cup D_2 \cup (E \cap Q_2)$ is the scheme theoretic base locus of the linear system $|\mathcal{I}_{D_1 \cup D_2 \cup (E \cap Q_2), Q_2}(3)|$ on $Q_2$. We call $(1, 0)$ the system of lines of $Q_2$ contained $D_1$ and $D_2$. Since $Y$ is nodal, $D_1 \cup D_2 \cup (E \cap Q_2) = D_1 \cup D_2 \cup Z$ with $\deg(Z) = 4$ and $E \cap H = \{A_1, A_2\} \cup Z$. Since $\mathcal{I}_{D_1 \cup D_2, Q_2}(3) \cong \mathcal{O}_{Q_2}(1, 3)$, it is sufficient that $Z$ is not contained in a line. Since $\mathcal{I}_E(3)$ is spanned, no line contains a degree 4 subscheme of $E$. $\square$

**Lemma 5.8.** *Let $Y = E \cup D_1 \cup D_2$ as above. For general $Y$ we have $h^1(\mathcal{I}_Y(3)) = 0$.*

*Proof.* Let $M \subset \mathbb{P}^4$ be the hyperplane spanned by the lines $D_1$ and $D_2$. Set $Q' := Q \cap M$. Look at the Castelnuovo exact sequence

$$0 \longrightarrow \mathcal{I}_E(2) \longrightarrow \mathcal{I}_Y(3) \longrightarrow \mathcal{I}_{D_1 \cup D_2 \cap (E \cap Q'), Q'}(3) \longrightarrow 0 \tag{6}$$

Since $h^1(\mathcal{I}_E(2)) = 0$ (Lemma 5.4), the exact sequence (6) shows that it is sufficient to prove that $h^1(Q', \mathcal{I}_{D_1 \cup D_2 \cap (E \cap Q'), Q'}(3)) = 0$. Since the integral quadric surface $Q'$ contains $D_1$ and $D_2$ and $D_1 \cap D_2 = \emptyset$, $Q'$ is a smooth quadric surface. Call $(1,0)$ the ruling of $Q'$ containing $D_1$ and $D_2$. For general $D_1$ and $D_2$ the hyperplane $M$ is not tangent to $E$ either at $A_1$ or at $A_2$. Hence the scheme $E \cap Q'$ is the disjoint union of $\{A_1, A_2\}$ (with its reduced structure) and a degree 4 scheme $Z$. Since $D_i \cap E = \{A_i\}$, we have $Z \cap (D_1 \cup D_2) = \emptyset$. Hence it is sufficient to prove $h^1(Q', \mathcal{I}_{Z,Q'}(1,3)) = 0$. Since $\deg(Z) = 4$, it is easy to check that $h^1(Q', \mathcal{I}_{Z,Q'}(1,3)) > 0$ if and only if there is a scheme $B \subset Z$ with $\deg(B) = 3$ and $B$ in a line of type $(1,0)$ on $Q'$. To exclude the existence of such a scheme $B$ it is sufficient to find $E$ without a two-dimensional family of trisecant lines (move $D_1$ and $D_2$). □

*Proof of Proposition 5.3.* By the Serre correspondence it is sufficient to find a smooth elliptic curve $C \subset Q$ such that $\mathcal{I}_C(3)$ is spanned, $h^1(\mathcal{I}_C(t)) = 0$ for all $t \geq 3$ and $h^0(\mathcal{I}_C(2)) = 0$ (indeed, the last condition implies $h^0(\mathcal{I}_C(t)) = 0$ for all $t \leq 1$ and hence $h^1(\mathcal{I}_C(2)) = 2$, $h^1(\mathcal{I}_C(1)) = 3$ and $h^1(\mathcal{I}_C(t)) = 0$ for all $t \leq 0$ by the Riemann-Roch theorem). By semicontinuity it is sufficient to find $Y$ with the same properties. There is $Y$ with $\mathcal{I}_Y(3)$ spanned (Lemma 5.7) and with $h^1(\mathcal{I}_Y(3)) = 0$ (Lemma 5.7). Castelnuovo-Mumford's lemma implies $h^1(\mathcal{I}_Y(t)) = 0$ for all $t \geq 4$. Since $Y = E \cup D_1 \cup D_2$ with $h^0(\mathcal{I}_E(2)) = 0$ (Lemma 5.4), we have $h^0(\mathcal{I}_Y(2)) = 0$. □

**Remark 5.9.** For a vector bundle $\mathcal{E}$ in $\mathfrak{M}(3,8)$ that fits into the sequence:

$$0 \to \mathcal{O}_Q(1) \to \mathcal{E} \to \mathcal{I}_Z(2) \to 0,$$

where $Z$ is the disjoint union of 4 lines, we can easily compute that $h^2(\mathcal{E}nd(\mathcal{E})) = 0$. It implies that $\mathcal{E}$ is a smooth point of $\mathfrak{M}(3,8)$ and the dimension of $\mathfrak{M}(3,8)$ is 18 since such bundles form a 15-dimensional subvariety of $\mathfrak{M}(3,8)$ and $\chi(\mathcal{E}nd(\mathcal{E})) = -17$.

## 6. Case $(c_1, c_2) = (3,9)$

Assume that $c_2(\mathcal{E}) = 9$. Again let $\mathcal{F} = \mathcal{E}(-2)$. We can compute:

$$\begin{cases} H^1(\mathcal{F}(t)) = 0 \text{ for } t \leq -2 \\ \chi(\mathcal{F} \otimes \mathcal{F}^\vee) = -23 \\ \chi(\mathcal{F}) = -\chi(\mathcal{F}(-2)) = -4 \\ \chi(\mathcal{F}(1)) = -\chi(\mathcal{F}(-3)) = -4 \\ \chi(\mathcal{F}(-1)) = 0 \\ \chi(\mathcal{F}(2)) = 4 \end{cases}$$

Then the cohomology table for $\mathcal{F} = \mathcal{E}(-2)$ is as follows:

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 4 | c | 0 | 0 | 0 |
| 0 | 0 | c | 4 | 4 | $b$ |
| 0 | 0 | 0 | 0 | 0 | $a$ |
| -3 | -2 | -1 | 0 | 1 | 2 |

with $a-b=4$. If $\{k_1 \leq \cdots \leq k_4\}$ is the spectrum of $\mathcal{F}$, then the possibility is either $\{-2,-1,-1,0\}$ or $\{-1,-1,-1,-1\}$. It implies that $H^1(\mathcal{E}(-3)) = 0$ or 1, i.e. $c = 0$ or 1 in the table. Thus $h^0(\mathcal{O}_C)$ is either 1 or 2. Since $\omega_C \simeq \mathcal{O}_C$, so $C$ is either an irreducible elliptic curve of degree 9 or consists of two irreducible elliptic curves. Note that $C$ cannot have a plane cubic curve as its component.

**Lemma 6.1.** *$C$ is an irreducible smooth elliptic curve of degree 9.*

*Proof.* Assume that $C = C_1 \sqcup C_2$ with $C_i$ smooth elliptic, $\deg(C_1) = 4$ and $\deg(C_2) = 5$. The curve $C_1$ is a contained in a hyperplane section $J$, which may be a cone. Even if $J$ is not smooth, $C_1$ is a complete intersection of $J$ with a quadric surface in the linear span $\langle J \rangle \cong \mathbb{P}^3$. So in $J \cap (C_1 \cup C_2)$ we have $C_1$ and a degree 5 scheme $J \cap C_2$; the only possibility to have $\mathcal{I}_{C_1 \cup C_2}(3)$ spanned is that $\mathcal{I}_{J\cap C_2, J}(1)$ is spanned, which is absurd. □

In particular, we have $c = 0$ in the cohomology table of $\mathcal{F}$. Similarly as in the proposition 5.1, we can show that $\mathcal{E}$ is the cohomology of a monad:

$$0 \to \mathcal{O}_Q(1)^{\oplus 4} \to \Sigma(1)^{\oplus 5} \to \mathcal{O}_Q(2)^{\oplus 4} \to 0.$$

Now we will prove the existence of a smooth elliptic curve $C \subset Q$ such that $\deg(C) = 9$ and $h^1(\mathcal{I}_C(3)) = 0$. Since $3 \cdot 9 = 27 = h^0(\mathcal{O}_Q(3)) - 3$, we have $h^1(\mathcal{I}_C(3)) = 0$ if and only if $h^0(\mathcal{I}_C(3)) = 3$. The latter condition obviously implies $h^0(\mathcal{I}_C(2)) = 0$. Hence proving the existence of $C$ also proves the existence of a rank two vector bundle $\mathcal{E}$ on $Q$ with $(c_1, c_2) = (3, 9)$, $h^0(\mathcal{E}) = 4$, $h^1(\mathcal{E}) = 0$, $h^0(\mathcal{E}(-1)) = 0$ (and hence $\mathcal{E}$ is stable).

**Lemma 6.2.** *There is a smooth elliptic curve $A \subset Q$ such that* $\deg(A) = 7$ *and $h^0(\mathcal{I}_A(2)) = 0$.*

*Proof.* We start with a smooth hyperplane section $Q_2$ of $Q$ and a smooth elliptic curve $B \subset Q_2$ of type $(2,2)$. Fix 2 general points $P_1, P_2 \in B$. The union $U(P_i)$ of all lines in $Q$ passing through $P_i$ is the quadric cone $T_{P_i}(Q) \cap Q$ of the tangent space $T_{P_i}(Q) \cong \mathbb{P}^3$ of $Q$ at $P_i$. Hence we may find a line $D_1 \subset Q$ such that $P_1 \in D_1$ and a smooth conic $D_2 \subset Q$ such that $D_2 \cap Q_2$ is $P_2$ and a point not in $B$. For general $D_i$ we may also assume $D_1 \cap D_2 = \emptyset$ and $D_i \nsubseteq Q_2$. For general $D_1, D_2$ we may also assume that no hyperplane section of $Q$ contains $D_1 \cup D_2$. Set $Y := B \cup D_1 \cup D_2$. Since $Q_2$ is a hyperplane section of $Q$, $D_i \nsubseteq Q_2$ and $B \subset Q_2$, then $Q_2 \cap D_i = \{P_i\}$ and $D_i$ is not the tangent line of $B$ at $P_i$. Hence $Y$ is a nodal connected curve of degree 7 and with arithmetic genus 1.

The reduced curve $Y$ is locally a complete intersection and hence its normal sheaf $\mathcal{N}_Y$ is a rank 2 vector bundle of degree $\deg(TQ|_Y) = 21$. Since $Y = B \cup (D_1 \cup D_2)$ has nodes only at $P_1, P_2$, the vector bundles $\mathcal{N}_Y|_B$ are obtained from $\mathcal{N}_B$ making 2 positive elementary transformations, each of them at one of the points $P_i$. Since $\mathcal{N}_B$ is a direct sum of a line bundle of degree 4 and a line bundle of degree 8, we have $h^1(B, \mathcal{N}_B) = 0$. Hence $h^1(B, \mathcal{N}_Y|_B) = 0$. The normal bundle $\mathcal{N}_{D_1}$ is a direct sum of a degree 0 line bundle and a degree 1 line bundle. The normal bundle $\mathcal{N}_{D_2}$ is a direct sum of a degree 2 line bundle and a degree 4 line bundle. Hence for every vector bundle $\mathcal{F}$ on $D_i$ obtained from $\mathcal{N}_{D_1 \cup D_2}$ making 2 negative elementary transformations, each of them at a different point $P_1, P_2$, has no factor of degree $\leq -2$. Hence $h^1(D_i, \mathcal{F}) = 0$. Hence $h^1(D_1 \cup D_2, \mathcal{G}) = 0$, where $\mathcal{G}$ is the vector bundle obtained from $\mathcal{N}_{D_1 \cup D_2}$ making the 2 negative elementary transformations at $P_1$ and $P_2$ associated to the tangent lines of $B$ at these points. Hence $h^1(Y, \mathcal{N}_Y) = 0$ and $Y$ is smoothable inside $Q$ (use [7], Theorem 4.1, for $Q$ instead of the smooth 3-fold $\mathbb{P}^3$). By semicontinuity to prove Lemma 6.2 it is sufficient to prove $h^0(\mathcal{I}_Y(2)) = 0$. Assume $h^0(\mathcal{I}_Y(2)) > 0$ and take $\Delta \in |\mathcal{I}_Y(2)|$. Since $Q_2 \cap Y$ contains $B$ and a point of $D_2 \cap (Q_2 \setminus B)$, we have $Q_2 \subset \Delta$, i.e. $\Delta = Q_2 \cup Q'$ for some hyperplane section $Q'$ of $Q$. Since neither $D_1$ nor $D_2$ is contained in $Q_2$, we get $D_1 \cup D_2 \subset Q'$, contradicting our choice of $D_1 \cup D_2$. $\square$

**Lemma 6.3.** *There is a smooth elliptic curve $C \subset Q$ such that* $\deg(C) = 9$, $h^0(\mathcal{I}_C(3)) = 3$ *and* $h^1(\mathcal{I}_C(3)) = 0$.

*Proof.* Let $C \subset Q$ be any smooth elliptic curve of degree 9. Since $h^0(\mathcal{O}_C(3)) = 27 = h^0(\mathcal{O}_Q(3)) - 3$, we have $h^1(\mathcal{I}_C(3)) = h^0(\mathcal{I}_C(3)) - 3$. Hence it is sufficient to prove the existence of a smooth elliptic curve $C$ with $\deg(C) = 9$ and $h^0(\mathcal{I}_C(3)) = 3$. Let $A \subset Q$ be a smooth elliptic curve such that $\deg(A) = 7$ and $h^0(\mathcal{I}_A(2)) = 0$. Fix two general $P_1, P_2 \in A$ (Lemma 6.2). The union $U(P_i)$ of all lines in $Q$ passing through $P_i$ is the quadric cone $T_{P_i}(Q) \cap Q$ of the tangent space $T_{P_i}(Q)$ of $Q$ at $P_i$. Hence we may find lines $D_i \subset Q$, $i = 1, 2$, such that $P_i \in D_i$, $D_i$ is not the tangent line to $A$ at $P_i$, $D_1 \cap D_2 = \emptyset$ and $D_i \cap A = \{P_i\}$. Hence $Y := A \cup D_1 \cup D_2$ is a connected nodal curve with degree 9 and arithmetic genus 1. As in the proof of Lemma 6.2 we see that $Y$ is smoothable inside $Q$. Hence by semicontinuity to prove Lemma 6.3 it is sufficient to prove $h^0(\mathcal{I}_Y(3)) = 3$. Let $H \subset \mathbb{P}^4$ be the hyperplane spanned by $D_1 \cup D_2$. Set $Q' := Q \cap H$. Since $Q'$ contains the disjoint lines $D_1, D_2$, $Q'$ is a smooth quadric surface. Call $(1,0)$ the ruling of $Q'$ containing $D_1$ and $D_2$. Fix general $O_1, O_2, O_3 \in Q'$. Since $h^0(\mathcal{I}_Y(3)) \geq 3$ by the Riemann-Roch theorem, to prove $h^0(\mathcal{I}_Y(3)) = 3$ it is sufficient to prove $h^0(\mathcal{I}_{Y \cup \{O_1, O_2, O_3\}}(3)) = 0$. Assume $h^0(\mathcal{I}_{Y \cup \{O_1, O_2, O_3\}}(3)) > 0$ and take $\Delta \in |\mathcal{I}_{Y \cup \{O_1, O_2, O_3\}}(3)|$. For general $D_1$ and $D_2$ we may also assume that $H$ is not tangent to $A$ neither at $P_1$ nor at $P_2$. Hence the scheme $A \cap Q'$ is the disjoint union of $P_1$, $P_2$, and a degree 5 scheme, $Z$, such that $Z \cap (D_1 \cup D_2) = \emptyset$. First assume $Q' \subset \Delta$. Hence $\Delta = Q' \cup T$ with $T$ a quadric hypersurface

of $Q$. Since $Q' \cap A$ is a finite set, we get $A \subset T$. Hence $h^0(\mathcal{I}_A(2)) > 0$, a contradiction. Hence $\Delta' := \Delta \cap Q'$ is a divisor of type $(3,3)$ of $Q'$ containing $D_1 \cup D_2$. Set $J := \Delta' - D_1 - D_2$. $J$ is a divisor of type $(1,3)$ on $Q'$ containing $Z$, $O_1$, $O_2$ and $O_3$. Since the points $O_i$ are general in $Q'$, to get a contradiction (and hence to prove the lemma) it is sufficient to prove $h^0(Q', \mathcal{I}_Z(1,3)) = 3$, i.e. $h^1(\mathcal{I}_Z(1,3)) = 0$. Fix a smooth $D \in |\mathcal{O}_{Q'}(1,1)|$ and take a general hyperplane section $T$ of $Q$ with $T \cap T' = D$. $T$ is a smooth quadric surface. We may specialize $A$ to a curve $Y' := A' \cup L_1 \cup L_2 \cup L_3$ with $A'$ a smooth curve of type $(2,2)$, each $L_i$ a line intersecting transversally $A'$ and $P_i \in L_i$, $i = 1,2$. For general $Y'$ we have $Y \cap Q' = Z' \cup \{P_1, P_2\}$ with $\sharp(Z' \cap D) = 4$. By semicontinuity it is sufficient to prove $h^1(Q', \mathcal{I}_{Z'}(1,3)) = 0$. Since $\sharp(Z' \setminus Z' \cap D) = 1$, we have $h^1(Q', \mathcal{I}_{Z' \setminus Z' \cap D; Q'}(0,2)) = 0$. $D$ is a smooth rational curve and $\deg(\mathcal{O}_D(1,3)) = 4$. Hence $h^1(D, \mathcal{I}_{D \cap Z', S}(1,3)) = 0$. From the exact sequence on $Q'$:

$$0 \longrightarrow \mathcal{I}_{Z' \setminus Z' \cap D, Q'}(0,2) \longrightarrow \mathcal{I}_{Z', Q'}(1,3) \longrightarrow \mathcal{I}_{D \cap Z', D}(1,3) \longrightarrow 0$$

we get $h^1(Q', \mathcal{I}_{Z'}(1,3)) = 0$, concluding the proof. $\square$

Let us start with the smooth elliptic curve $C$ given by Lemma 6.3 ; hence $h^1(\mathcal{I}_C(3)) = 0$ and $h^0(\mathcal{I}_C(3)) = 3$. Since $h^0(\mathcal{I}_C(3)) \geq 2$, there are at least two degree 3 hypersurfaces $M_1, M_2$ containing $C$. Since $h^0(\mathcal{I}_C(2)) = 0$, the scheme-theoretic intersection $X := M_1 \cap M_2$ has dimension 1, i.e. it is a complete intersection. By the Liaison induced by $X$, we have

$$0 \longrightarrow \mathcal{I}_X(3) \longrightarrow \mathcal{I}_C(3) \longrightarrow \omega_Y \longrightarrow 0. \tag{7}$$

We need to prove that $\omega_Y$ is spanned. Since $X$ is a complete intersection of $M_1$ and $M_2$, we have $h^0(\mathcal{I}_X(3)) = 2$ and $h^1(\mathcal{I}_X(3)) = 0$. Since $h^0(\mathcal{I}_C(3)) = 3$, the sequence (7) gives $h^0(\omega_Y) = 1$. We have $h^2(\mathcal{I}_C(3)) = h^1(\mathcal{O}_C(3)) = 0$ and $h^2(\mathcal{I}_X(3)) = h^1(\mathcal{O}_X(3)) = 1$ since $\omega_X \cong \mathcal{O}_X(3)$ by the adjunction formula. Since $h^1(\mathcal{I}_C(3)) = 0$ we get $h^1(\omega_Y) = 1$. Hence $h^0(\mathcal{O}_Y) = 1$ by the duality of locally Cohen-Macaulay projective schemes. Since $p_a(Y) = 1$ to get that $\omega_Y$ is trivial and hence that $\omega_Y$ is spanned, it is sufficient to prove that (at least for certain $C$) it is spanned outside finitely many points. Since $\omega_Y$ is a quotient of $\mathcal{I}_C(3)$, it is spanned at all points at which $\mathcal{I}_C(3)$ is spanned. Hence it is sufficient to find $C$ with the additional condition that $\mathcal{I}_C(3)$ is spanned outside finitely many points. This is the Lemma 6.4 below.

**Lemma 6.4.** *There is a smooth elliptic curve $C \subset Q$ such that $\deg(C) = 9$, $h^0(\mathcal{I}_C(3)) = 3$, $h^1(\mathcal{I}_C(3)) = 0$ and such that $\mathcal{I}_C(3)$ is spanned outside finitely many points.*

*Proof.* By semicontinuity it is sufficient to find $Y = B \cup D_1 \cup D_2$ as in the proof of Lemma 6.3 with the additional property that the base locus of $\mathcal{I}_Y(3)$ is finite. Fix $B$ satisfying the thesis of Lemma 6.2. Let $H \subset \mathbb{P}^4$ be a general hyperplane. By the Uniform Position Principle ([2], pp. 112–113) the scheme $B \cap H$ is formed by 7 points in uniform position and we call

$A_1, A_2$ two of them. Moreover, the monodromy of the generic hyperplane section is the full transite group ([2], p. 112). Hence for general $H$ we may assume that no two of the points of $A \cap H$ are contained in a line in $H$. Set $Q' := Q \cap H$. For general $H$ the scheme $Q'$ is a smooth quadric surface. Fix one of the system of lines of $Q'$, say $(1,0)$, and call $D_i$ the line of type $(1,0)$ of $Q'$ containing $A_i$. Set $S := B \cap Q' \setminus \{A_1, A_2\}$. Notice that $\sharp(S) = 5$ and $S \cap D_i = \emptyset$. Set $Y := B \cup D_1 \cup D_2$. The proof of Lemma 6.3 gives $h^1(\mathcal{I}_Y(3)) = 0$ and $h^0(\mathcal{I}_Y(3)) = 3$. Let $\mathcal{B}$ denote the base locus of $\mathcal{I}_Y(3)$. It is sufficient to prove that $\mathcal{B} \cap Q' = \emptyset$. Since $h^i(\mathcal{I}_B(2)) = 0$, $i = 0, 1$, we saw in the proof of Lemma 6.3 that the restriction of $|\mathcal{I}_Y(3)|$ to $Q'$ is given by all forms $D_1 \cup D_2 \cup T$ with $T \in |\mathcal{I}_S(1,3)|$ and $h^0(Q', \mathcal{I}_S(1,3)) = 3$. Since $S$ is in uniform position, $T$ is a general element of $|\mathcal{I}_S(1,3)|$. Fix two general $T_1, T_2 \in |\mathcal{I}_S(1,3)|$. Since $T_1$ is irreducible, the scheme $T_1 \cap T_2$ is zero-dimensional. We have $\deg(T_1 \cap T_2) = 3 + 3 = 6$. Since $S \subset T_1 \cap T_2$ and $h^0(Q', \mathcal{I}_S(1,3)) = 3$, we get that $\mathcal{I}_S(1,3)$ is a spanned sheaf of $Q'$. Since $S \cap (D_1 \cup D_2) = \emptyset$, we also get that the scheme $Y \cap Q'$ is the intersection with $Q'$ of all elements of $|\mathcal{I}_Y(3)|$. Hence $\mathcal{B} \cap Q' = \emptyset$. Hence $\mathcal{B}$ is supported by finitely many points. $\square$

**Remark 6.5.** Similarly as in the case $c_1 = 8$, for a vector bundle $\mathcal{E} \in \mathfrak{M}(3,9)$ that fits into the sequence:

$$0 \to \mathcal{O}_Q(1) \to \mathcal{E} \to \mathcal{I}_Z(2) \to 0,$$

where $Z$ is the disjoint union of 5 lines, we can easily compute that $h^2(\mathcal{E}nd(\mathcal{E})) = 0$. It implies that $\mathcal{E}$ is a smooth point of $\mathfrak{M}(3,9)$ and the dimension of $\mathfrak{M}(3,9)$ is 24 since such bundles form a 19-dimensional subvariety of $\mathfrak{M}(3,9)$ and $\chi(\mathcal{E}nd(\mathcal{E})) = -23$.

As an automatic consequence from the classification, we observe that if $\mathcal{E}$ is a rank two globally generated vector bundle on $Q$ with $c_1 = 3$, then we have $c_1(\mathcal{E}(-2)) = -1$ and $H^1(\mathcal{E}(-3)) = 0$. Thus we have the following:

**Corollary 6.6.** *Every rank two globally generated vector bundle on $Q$ with $c_1 = 3$, is an odd instanton (see [6]) up to twist.*

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